QUANTUM UNIQUE ERGODICITY ON LOCALLY SYMMETRIC SPACES: THE DEGENERATE LIFT

LIOR SILBERMAN

ABSTRACT. Given a measure $\bar{\mu}_\infty$ on a locally symmetric space $Y = \Gamma\backslash G/K$, obtained as a weak-* limit of probability measures associated to eigenfunctions of the ring of invariant differential operators, we construct a measure μ_∞ on the homogeneous space $X = \Gamma\backslash G$ which lifts $\bar{\mu}_\infty$ and which is invariant by a connected subgroup $A_1 \subset A$ of positive dimension, where $G = NAK$ is an Iwasawa decomposition. If the functions are, in addition, eigenfunctions of the Hecke operators, then μ_∞ is also the limit of measures associated to Hecke eigenfunctions on X. This generalizes previous results of the author and A. Venkatesh to the case of "degenerate" limiting spectral parameters.

1. INTRODUCTION

In the work of the author with A. Venkatesh [14] we investigated the asymptotic behaviour of eigenfunctions on high-rank locally symmetric spaces, under the assumption that the spectral parameters (see below) were *non-degenerate*, in that their imaginary parts were located away from the walls of the Weyl chamber (in particular, this forced the spectral parameters to lie on the unitary axis). This paper removes this assumption, at the cost of a weaker invariance statement for the limiting measures. The main extra ingredient is a simple calculation in the "compact" model of induced representation for semisimple Lie groups.

1.1. **The problem of Quantum Unique Ergodicity; statement of the result.** Let Y be a (compact) Riemannian manifold. To a non-zero eigenfunction ψ_n of the Laplace-Beltrami operator $\triangle$ with eigenvalue $-\lambda_n$ we attach the probability measure

$$\bar{\mu}_n(\varphi) = \frac{1}{\|\psi_n\|^2} \int_Y |\psi_n(y)|^2 \varphi(y) dy.$$

Classifying the possible limits (in the weak-* sense) of sequences $\{\bar{\mu}_n\}_{n=1}^\infty$ where $\lambda_n \to \infty$ is known as the problem of "Quantum Unique Ergodicity" (specifically, "QUE on Y"). Nearly all attacks on this problem begin by associating to each measure $\bar{\mu}_n$ a distribution ("microlocal lift") μ_n on the unit cotangent bundle S^*Y which projects to $\bar{\mu}_n$ on Y, in such a way that any weak-* limit of the μ_n is a probability measure, invariant under the geodesic flow on S^*Y. This construction (due to Schnirel'man, Zelditch and Colin de Verdière, [15, 17, 5]) leads to a reformulation of the problem ("QUE on S^*Y"), where one seeks to classify the weak-* limits of sequences such as $\{\mu_n\}_{n=1}^\infty$. Now results from dynamical systems concerning measures invariant under the geodesic flow can be brought to bear. In particular, under the very general assumption that the geodesic flow on S^*Y is ergodic, it was shown by these authors that the Riemannian volume measure on S^*Y

is always a limit measure for some sequence of eigenfunctions (hence its projection, the Riemannian volume on Y, is always a limit of a sequence of measures $\bar{\mu}_n$). The most spectacular realization of this approach to QUE is in the work of Lindenstrauss [9]. There it is shown that on congruence hyperbolic surfaces and for eigenfunctions ψ_n which are also eigenfunctions of the so-called Hecke operators the Riemannian volume is the *only* limiting measure[1]. In fact, Rudnick-Sarnak [12] conjecture that this phenomenon (uniqueness of the limit) holds for all manifolds Y of (possibly variable) negative sectional curvature. Results in that level of generality have also appeared recently, starting with the breakthrough of [1].

In this paper we consider a technical aspect of the problem on locally symmetric spaces $Y = \Gamma\backslash G/K$ of non-compact type. Here G is a semisimple Lie group with finite center, K a maximal compact subgroup and $\Gamma < G$ a lattice (thus Y is of finite volume but not necessarily compact). On such spaces there is a natural commutative algebra of differential operators containing the Laplace-Beltrami operator, and it is better to consider joint eigenfunctions of this algebra. This is the algebra of G-invariant differential operators on G/K, which may be identified with the center of the universal enveloping algebra of the Lie algebra of G. Accordingly, let $\psi_n \in L^2(Y)$ be joint eigenfunctions of this algebra. The approach of microlocal analysis applies to this setting as well (see [2]), lifting measures to distributions on S^*Y, but in fact limits of these measures are supported on singular subsets there, isomorphic to submanifolds of the form $\Gamma\backslash G/M_1$ for compact subgroups M_1. Here we directly construct a lift to this space. Moreover, in the congruence setting it is desirable to have the lift be manifestly equivariant with respect to the action of the Hecke algebra. In the paper [14] this was done under a genericity assumption ("non-degeneracy") – that the sequence of spectral parameters $\nu_n \in \mathfrak{a}^*_{\mathbb{C}}$ (here $\mathfrak{a} = \mathrm{Lie}(A)$ where $G = NAK$ is an Iwasawa decomposition) associated to the ψ_n be contained in a proper subcone of the open Weyl chamber in $i\mathfrak{a}^*_{\mathbb{R}}$. Under that assumption, and weak-* limit $\bar{\mu}_\infty$ of a sequence as above was seen to be the projection of an A-invariant positive measure μ_∞ on X. In this paper the non-degeneracy assumption is removed, giving our main result:

Theorem 1. *Assume $\bar{\mu}_n \xrightarrow[n\to\infty]{wk\text{-}*} \bar{\mu}_\infty$. Then there exists a non-trivial connected subgroup $A_1 \subset A$ and an A_1-invariant positive measure σ_∞ on X projecting to $\bar{\mu}_\infty$.*

In more detail, let $C^\infty_c(X)_K$ be the space of right K-finite smooth functions of compact support on $X = \Gamma\backslash G$. By a distribution *we shall mean an element of its algebraic dual. Then, after passing to a subsequence, we obtain distributions $\mu_n \in C^\infty_c(X)'_K$ and functions $\tilde{\psi}_n \in L^2(X)$ such that:*

   (1) *(Lift) The distributions μ_n project to the measures $\bar{\mu}_n$ on Y. In other words, for $\varphi \in C^\infty_c(Y)$ we have $\mu_n(\varphi) = \bar{\mu}_n(\varphi)$.*
   (2) *Let σ_n be the measure on X such that $d\sigma_n(x) = |\tilde{\psi}_n(x)|^2 \, dx$. Then:*
      (a) *(Positivity) $\{\sigma_n\}_{n=1}^\infty$ converges weak-* to a measure σ_∞ on X, necessarily a positive measure of total mass ≤ 1.*
      (b) *(Consistency) For any $\varphi \in C^\infty_c(X)_K$, $|\sigma_n(\varphi) - \mu_n(\varphi)| \to 0$ as $n \to \infty$.*
   (3) *(Invariance) Let the normalized spectral parameters[2] $\tilde{\nu}_n$ converge to a limiting parameter $\tilde{\nu}_\infty$ in the closed positive Weyl chamber of $i\mathfrak{a}^*_{\mathbb{R}}$. Then μ_∞ is invariant by $A_1 Z_K(A_1)$, where $A_1 \subset A$ is the set of elements fixed by the stabilizer $W_1 = \mathrm{Stab}_W(\tilde{\nu}_\infty)$.*

---

[1]For non-compact surfaces this statement requires the result of [16].

[2]For G simple, these are $\frac{\nu_n}{\|\nu_n\|}$. For G semisimple see the discussion in [14, §5.1]

(4) (Equivariance) $\tilde{\psi}_n$ belong to the irreducible subrepresentation of G in $L^2(Y)$ generated by ψ_n. In particular, if $\mathcal{H}$ is a commutative algebra of bounded operators on $L^2(X)$ which commute with the G-action and ψ_n is a joint eigenfunction of $\mathcal{H}$ then so is $\tilde{\psi}_n$, with the same eigenvalues.

1.2. **Sketch of the proof.** As can be expected, we shall trace a path very similar to that of the previous work. Choose a pair of functions, one from the irreducible subrepresentation of $L^2(X)$ generated by ψ_n and one from its dual. Now integrating a function on X against the product of these two functions defines a measure there ($\bar{\mu}_n$ is a special case of this construction), and we will study limits of this larger family of measures. We construct an asymptotic calculus for these measures by uniformizing the representation via the compact picture of a principal series representations induced from a potentially non-unitary character. A prerequisite for taking limits in this setting is the following *a-priori* bound on these measures w.r.t. the uniformization, which is the key ingredient that was not available during the writing of [14].

Theorem 2. *Let $(\pi, V_\pi) \in \hat{G}$ be spherical, and let $R\colon (I_\nu, V_K) \to (\pi, V_\pi)$ be an[3] intertwining operator with the real part of $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ in the closed positive chamber $\mathcal{C}$, normalized such that $\|R(\varphi_0)\|_{V_\pi} = 1$, where $\varphi_0 \in V_K$ is the constant function 1. We then have $\|R(f)\|_{V_\pi} \leq \|f\|_{L^2(K)}$ for any $f \in V_K$.*

Surprisingly, we could not find this useful fact in the literature. It is proved in Section 3 as a consequence of the rationality of K-finite matrix coefficients by bounding the analytical continuation of the normalized intertwining operators $\tilde{A}(\nu; w)\colon (I_\nu, V_K) \to (I_{w\nu}, V_K)$ associated to elements w of the Weyl group W.

With this bound in hand we extend the asymptotic calculus of [14] to our setting. We construct the distributions μ_n in Section 4.1 (see Definition 18). Integration by parts gives the measures σ_n and establishes their properties (Corollaries 24 and 25). Finally, in Section 4.3 we obtain the desired invariance property.

1.3. **A measure rigidity problem on locally symmetric spaces.** An introduction to the relations between the general problem of Quantum Unique Ergodicity and the cases of manifolds of negative curvature and locally symmetric spaces of non-positive curvature may be found in the paper [14]. We consider here only the latter case, where again we have the Conjecture

Conjecture 3. *(Sarnak) The sequence $\{\bar{\mu}_n\}_{n=1}^\infty$ converges weak-* to the normalized volume measure $\frac{d\operatorname{vol}_Y}{\operatorname{vol}(Y)}$.*

We recall the strategy pioneered by Lindenstrauss, which applies for a lattice Γ for which there exists a large algebra $\mathcal{H}$ of bounded normal operators on $L^2(X)$, commuting with the G-action. We then consider a sequence of joint eigenfunctions of both the differential operators and of $\mathcal{H}$, and assume the associated measures $\bar{\mu}_n$ converge to a measure $\bar{\mu}_\infty$.

(1) *Lift*: Passing to a subsequence, lift $\bar{\mu}_\infty$ to a positive measure μ_∞ on X which projects to $\bar{\mu}_\infty$ under averaging by K and is invariant under a subgroup $H < G$, in a way which respects the $\mathcal{H}$-action.
(2) *Extra smoothness*: Using the geometry of the action of $\mathcal{H}$, show that any measure μ_∞ thus obtained is not too singular (for example, that the dimension of its support must be strictly larger than that of H).

---

[3]Such R always exist.

    (3) *Measure rigidity*: Using classification results for H-invariant measures on X, show that the additional information of Step (2) forces μ_∞ to be a G-invariant measure on X.

The result of this paper extend Step (1) of the strategy to the degenerate case and the methods used for Step (2) in [4] and [13] only use the Hecke operators. Unfortunately, current higher-rank measure classification results (such as the one in [6], used for Step (3) in [13]) do not readily generalize to the case of A_1-invariant measures; in this context see the counter-example [[11]. However, we are not considering a general A_1-invariant measure, so the natural question from our point of the view is the following. It should be compared with Lindenstrauss's rank 1 measure classification Theorem [9, Thm. 1.1].

Problem 4. Let $\Gamma < G$ be a congruence lattice associated to a $\mathbb{Q}$-structure on G and let $A_1 \subset A$ be a non-trivial one-parameter subgroup fixed by a subgroup of the Weyl group. Let $\tilde{\psi}_n \in L^2(X)$ be eigenfunctions of the Hecke operators on $X = \Gamma \backslash G$ such that their associated probability measures σ_n converge weak-* to an A_1-invariant measure σ_∞. Is it true that σ_∞ is then a (continuous) linear combination of algebraic measures on X?

2. Notation and Preliminaries

2.1. **Structure theory – real groups.** Let G be a connected almost simple Lie group[4], $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. Let Θ be a Cartan involution for G, θ the differential of Θ at the identity and let $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ be the associated polar decomposition. We fix a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. Its dimension is the (real) rank $\mathrm{rk}\, G$.

The dual vector space to $\mathfrak{a}$ will be denoted $\mathfrak{a}_\mathbb{R}^*$, and will be distinguished from the complexification $\mathfrak{a}_\mathbb{C}^* \overset{\mathrm{def}}{=} \mathfrak{a}_\mathbb{R}^* \otimes_\mathbb{R} \mathbb{C}$. For $\alpha \in \mathfrak{a}_\mathbb{R}^*$ set $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{a} : [H, X] = \alpha(H)X\}$. Let $\Delta = \Delta(\mathfrak{g} : \mathfrak{a})$ denote the set of *roots* (the non-zero $\alpha \in \mathfrak{a}_\mathbb{R}^*$ such that $\mathfrak{g}_\alpha \neq 0$). Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, and $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ where $\mathfrak{m} = Z_\mathfrak{k}(\mathfrak{a})$. For $\alpha \in \mathfrak{a}_\mathbb{R}^*$ we set $p_\alpha = \dim \mathfrak{g}_\alpha$, $q_\alpha = \dim \mathfrak{g}_{2\alpha}$.

The Killing form B induces a positive-definite pairing $\langle X, Y \rangle = -B(X, \theta Y)$ on $\mathfrak{g}$ which remains non-degenerate when restricted to $\mathfrak{a}$. We identify $\mathfrak{a}$ and $\mathfrak{a}_\mathbb{R}^*$ via this pairing, giving us a non-degenerate pairing $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}_\mathbb{R}^*$ and letting $H_\alpha \in \mathfrak{a}$ denote the element corresponding to $\alpha \in \Delta$. With this Euclidean structure on $\mathfrak{a}_\mathbb{R}^*$ the subset Δ is a root system, and we denote its Weyl group by $W(\mathfrak{g} : \mathfrak{a})$. A root $\alpha \in \Delta$ is *reduced* if $\frac{1}{2}\alpha \notin \Delta$. The set of reduced roots $\Delta^\mathrm{r} \subset \Delta$ is a root system as well. To $w \in W$ we associate the subset $\Phi_w = \Delta^\mathrm{r} \cap \Delta^+ \cap w^{-1}\Delta^-$ of positive reduced roots β such that $w\beta$ is negative.

We fix a simple system $\Pi \subset \Delta$, giving us a notion of positivity, and let Δ^+ (Δ^-) denote the set of positive (negative) roots, $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} p_\alpha \alpha \in \mathfrak{a}_\mathbb{R}^*$. For $\beta \in \Delta^\mathrm{r}$ and $\nu \in \mathfrak{a}_\mathbb{C}^*$ we set $\nu_\beta = \frac{2\langle \nu, \beta \rangle}{\langle \beta, \beta \rangle}$. Then

$$\mathscr{C} = \left\{ \nu \in \mathfrak{a}_\mathbb{R}^* \mid \forall \beta \in \Pi : \nu_\beta > 0 \right\}$$

is the open positive Weyl chamber. Its closure will be denoted $\mathcal{C}$. We will also consider the open domain

$$\Omega = \mathscr{C} + i\mathfrak{a}_\mathbb{R}^* = \left\{ \nu \in \mathfrak{a}_\mathbb{C}^* \mid \Re(\nu) \in \mathscr{C} \right\}$$

and its closure $\bar{\Omega}$. More generally, for $w \in W$ we set

$$\mathscr{C}_w = \left\{ \nu \in \mathfrak{a}_\mathbb{R}^* \mid \forall \beta \in \Phi_w : \nu_\beta > 0 \right\}$$

leading in the same fashion to $\overline{\mathscr{C}}_w \subset \mathfrak{a}_\mathbb{R}^*$ and $\Omega_w \subset \bar{\Omega}_w \subset \mathfrak{a}_\mathbb{C}^*$.

---

[4]The results of this paper hold (with natural modifications) for reductive G. The details may be found in [14, §5.1].

Returning to the Lie algebra we set $\mathfrak{n} = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \theta\mathfrak{n} = \oplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$ and obtain the Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$. On the group level we set $K = \{g \in G \mid \Theta(g) = g\}$, $A = \exp\mathfrak{a}$, $N = \exp\mathfrak{n}$, $\bar{N} = \exp\bar{\mathfrak{n}}$. These are closed subgroups with Lie algebras $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \bar{\mathfrak{n}}$ respectively: K is a maximal compact subgroup, A a maximal diagonalizable subgroup and N a maximal unipotent subgroup. With these we have the Iwasawa decomposition $G = NAK$. Another important subgroup is $M = Z_K(\mathfrak{a})$ which normalizes $N, \bar{N}$. M is not necessarily connected, but $\mathfrak{m} = \mathrm{Lie}(M)$ holds, and $B = NAM$ is the Borel subgroup. The action of $W = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ on $\mathfrak{a}^*_{\mathbb{R}}$ gives an isomorphism of W and the algebraic Weyl group $W(\mathfrak{g} : \mathfrak{a})$ defined above.

Let dk be a probability Haar measure on K, da, dn Haar measures on A and N. Then $dn \cdot a^{2\rho} da \cdot dk$ is a Haar measure on G. The linear functional $f \mapsto \int_K f(k)dk$ on the space $\mathcal{F}^\rho = \{f \in C(G): f(nag) = a^{2\rho} f(g)\}$ is right G-invariant.

2.2. Complexification.
Let $\mathfrak{b}$ be a maximal torus in the compact Lie algebra $\mathfrak{m}$, $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$. Then $\mathfrak{h}$ is a maximal Abelian semisimple subalgebra of $\mathfrak{g}$, that is a Cartan subalgebra.

$\mathfrak{g}_{\mathbb{C}}$ is a complex semisimple Lie algebra of which $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra. We let $\Delta(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ denote the associated root system, $W(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ its Weyl group. The restriction of any $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ to $\mathfrak{a}$ is either a root of $\mathfrak{g}$ or zero. We fix a notion of positivity on $\Delta(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ compatible with our choice for $\Delta(\mathfrak{g} : \mathfrak{a})$, and let $\rho_{\mathfrak{h}} \in \mathfrak{h}^*_{\mathbb{C}}$ denote half the sum of the positive roots in $\Delta(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$. Once $\rho_{\mathfrak{h}}$ makes its appearance we shall use $\rho_{\mathfrak{a}}$ for ρ defined before.

The image of $N_K(\mathfrak{h})$ in $W(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ is $\tilde{W} = N_K(\mathfrak{h})/Z_M(\mathfrak{b})$, since any $k \in N_K(\mathfrak{h})$ must normalize $\mathfrak{a}$, $\mathfrak{b}$ separately.

Lemma 5. $W(\mathfrak{m} : \mathfrak{b}) \simeq N_M(\mathfrak{b})/Z_M(\mathfrak{b})$ *is normal in* $\tilde{W}$*; the quotient is naturally isomorphic to* $W(\mathfrak{g} : \mathfrak{a})$*.*

Proof. That $N_M(\mathfrak{b}) = N_K(\mathfrak{h}) \cap Z_K(\mathfrak{a})$ gives the first assertion, and shows that the quotient embeds in $W(\mathfrak{g} : \mathfrak{a})$ since $N_K(\mathfrak{h}) \subset N_K(\mathfrak{a})$. To show that the embedding is surjective let $w \in N_K(\mathfrak{a})$ and consider $\mathrm{Ad}(w)\mathfrak{b}$. This is the Lie algebra of a maximal torus of M ($\mathrm{Ad}(w)$ is an automorphism of M), hence conjugate to $\mathfrak{b}$ in M. In other words, there exists $m \in M$ such that $\mathrm{Ad}(w)\mathfrak{b} = \mathrm{Ad}(m)\mathfrak{b}$ and hence $m^{-1}w \in N_K(\mathfrak{b})$. This element also normalizes $\mathfrak{a}$, and hence $wM \in W$ has a representative in $N_K(\mathfrak{h})$. $\qquad\square$

Corollary 6. *Under the identification* $\mathfrak{a}^*_{\mathbb{C}} \simeq \{\nu \in \mathfrak{h}^*_{\mathbb{C}} \mid \nu \restriction_{\mathfrak{b}} \equiv 0\}$ *(dual to the identification* $\mathfrak{a} \simeq \mathfrak{h}/\mathfrak{b}$*) the group* $\tilde{W} \subset W(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ *acts on* $\mathfrak{a}^*_{\mathbb{C}}$ *via its quotient map to* W*.*

We let $U(\mathfrak{g}_{\mathbb{C}})$ denote the universal enveloping algebra of the complexification of $\mathfrak{g}$ (and similarly $U(\mathfrak{a}_{\mathbb{C}})$, $U(\mathfrak{n}_{\mathbb{C}})$ ...). In such an algebra we let $U(\mathfrak{g}_{\mathbb{C}})^{\leq d}$ denote the subspace generated by all (non-commutative) monomials in $\mathfrak{g}_{\mathbb{C}}$ of degree at most d.

2.3. Representation Theory.
For any continuous representation of K on a Fréchet space W, and $\tau \in \hat{K}$ we let W_τ denote the τ-isotypical subspace, and $W_K = \oplus_\tau W_\tau$ denote the (dense) subspace of K-finite vectors. We let $\hat{W}_K = \prod_\tau W_\tau$ denote the completion of W_K with respect to this decomposition. This is the space of formal sums $\sum_\tau w_\tau$ where $w_\tau \in W_\tau$. We endow $\hat{W}_K$ with the product topology, which is also the topology of convergence component-wise.

We specifically set $V = C(M\backslash K)$ with the right regular action of K and let V_K denote the space of K-finite vectors there. We also have $V_K = L^2(M\backslash K)_K$; $\hat{V}_K$ can be identified with the algebraic dual V'_K via the pairing $(f, \sum_\tau \phi_\tau) \mapsto \sum_\tau \int_{M\backslash K} f \cdot \phi_\tau$; the product topology is the weak-* topology. We let $\varphi_0 \in V_K$ denote the function everywhere equal to 1.

Definition 7. For $v \in \mathfrak{a}_{\mathbb{C}}^*$ let G act by the right regular representation on

$$\mathcal{F}^v = \left\{ \varphi \in C^\infty(G) | \varphi(namg) = a^{v+\rho} \varphi(g) \right\}.$$

This induces a $(\mathfrak{g}, K)$-module structure on the space of K-finite vectors $\mathcal{F}_K^v$. By the Iwasawa decomposition the restriction map $\mathcal{F}_K^v \to V_K$ is an isomorphism of algebraic representations of K, giving us a model (I_v, V_K) for F_K^v. Given $\Phi = \sum_\tau \phi_\tau \in \hat{V}_K$ and $X \in \mathfrak{g}$ we set $I_v(X)\Phi = \sum_\tau I_v(X)\phi_\tau$ (the τ'-component of the sum only has contribution from K-types appearing in the tensor product of τ' and the adjoint representation of K on $\mathfrak{g}$). Let $\bar{\mathbb{1}}$ denote the trivial representation of $(\mathfrak{g}, K)$ where the complex number z acts by multiplication by $\bar{z}$. Let $(\bar{I}_v, \hat{V}_K) = (I_v, \hat{V}_K) \otimes \bar{\mathbb{1}}$.

Notation 8. Let $(\mathcal{I}_v, \mathcal{V}_K)$ denote the $(\mathfrak{g}, K)$ module $(I_v \otimes \bar{I}_v, V_K \otimes \hat{V}_K)$.

Fact 9. *(Induced Representations)*

(1) *The pairing $(f, g) \mapsto \int_{M\backslash K} fg$ is a G-invariant pairing on $\mathcal{F}^v \otimes \mathcal{F}^{-v}$. Equivalently, $(f, g) \mapsto \int_{M\backslash K} f\bar{g}$ is an invariant Hermitian pairing between (I_v, V_K) and $(I_{-\bar{v}}, V_K)$. For $v \in i\mathfrak{a}_{\mathbb{R}}^*$ (the unitary axis) it follows that (I_v, V_K) is unitarizable, its invariant Hermitian form given by the standard pairing of $L^2(M\backslash K)$.*

(2) *The induced representation is irreducible for v lying in an open dense subset of $i\mathfrak{a}_{\mathbb{R}}^*$.*

(3) *Every irreducible spherical $(\mathfrak{g}, K)$-module (π, V_π) can be realized as a quotient via an intertwining operator $R: (I_v, V_K) \to (\pi, V_\pi)$, for some $v \in \mathcal{C}$.*

2.4. **Intertwining Operators.** Given $w \in W$ and $v \in \mathfrak{a}_{\mathbb{C}}^*$, we can uniquely extend any $\varphi \in V_K$ to an element of $\mathcal{F}^v$ (also denoted φ). For $v \in \mathscr{C}_w$ we can then define an endomorphism $A(v; w)$ of V_K by

$$(A(v; w)\varphi)(k) = \int_{\bar{N} \cap wNw^{-1}} \varphi(\bar{n}wk)d\bar{n}$$

(the integral converges absolutely in this case). It is easy to check that this operator intertwines the representations (I_v, V_K) and (I_{wv}, V_K) and is holomorphic in the domain Ω_w.

Fact 10. *(Intertwining operators)*

(1) [7, Prop. 60(i)] The operators $A(v; w)$ admit a meromorphic continuation to all of $\mathfrak{a}_{\mathbb{C}}^*$, intertwining the representations (I_v, V_K) and (I_{wv}, V_K). For $v \in i\mathfrak{a}_{\mathbb{R}}^*$ they are unitary operators.

(2) [8, §VII.5] For $v \in \mathfrak{a}_{\mathbb{C}}^*$ and $\beta \in \Delta$ set $v_\beta = \frac{2\langle v, \beta \rangle}{\langle \beta, \beta \rangle}$. For $w \in W$ set $\Phi_w = \left\{ \beta \in \Delta \setminus 2\Delta | \beta \in \Delta^+ \cap w^{-1}\Delta^- \right\}$. Then $A(v; w)\varphi_0 = r(v; w)\varphi_0$ where

$$r(v; w) = \prod_{\beta \in \Phi_w} \left[\frac{\Gamma(p_\beta + q_\beta)}{\Gamma(\frac{1}{2}(p_\beta + q_\beta))} \frac{\Gamma(\frac{1}{2}v_\beta)}{\Gamma(\frac{1}{2}(v_\beta + p_\beta))} \frac{\Gamma(\frac{1}{4}(v_\beta + p_\beta))}{\Gamma(\frac{1}{4}(v_\beta + p_\beta) + \frac{1}{2}q_\beta)} \right].$$

We set $\tilde{A}(v; w) = r^{-1}(v; w)A(v; w)$.

(3) [8, Ch. XVI]If the spherical representation (π, V_π) is unitarizable and realized as a quotient of (I_v, V_K) as before, these exists $w \in W$ with $w^2 = 1$ such that $wv = -\bar{v}$; further more $\Re(v)$ belongs to a fixed compact set.

(4) Conversely, let $w \in W$ satisfy $w^2 = 1$, and let $v \in \mathfrak{a}_{\mathbb{C}}^*$ such that $wv = -\bar{v}$. Then

$$(f, g) \mapsto \langle A(v; w)f, g \rangle_{L^2(K)}$$

defines a non-zero $(\mathfrak{g}, K)$-equivariant Hermitian pairing on (I_v, V_K); the subspace where the pairing vanishes is the kernel of $A(v; w)$ and the quotient is irreducible.

The quotient is unitarizable iff the pairing is semidefinite, and every unitary spherical representation arises this way.

(5) [3] For fixed $\varphi, \psi \in V_K$ the matrix coefficient

$$\nu \mapsto \left\langle \tilde{A}(\nu; w)\varphi, \psi \right\rangle_{L^2(K)}$$

is a rational function of ν where we identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}^{\dim \mathfrak{a}}$ via the map $\nu \mapsto (\nu(H_\alpha))_{\alpha \in \Pi}$.

Remark 11. Since V_K contains a unique copy of the trivial representation of K, we must have $A(\nu; w)\varphi_0 = r(\nu; w)\varphi_0$ for some meromorphic function $r(\nu; w)$. Showing the integral defining $r(\nu; w)$ converges absolutely for $\nu \in \mathscr{C}_w$ proves the absolute convergence claim above.

Since $r(\nu; w)$ does not vanish in open domain Ω_w, $\tilde{A}(\nu; w)$ cannot have zeroes or poles there.

3. INTERPOLATION BOUNDS ON INTERTWINING OPERATORS

Lemma 12. *Let $f \in \mathbb{C}(z)$ be a rational function of one variable. Assume that f is bounded on the line $\Re(z) = 0$ and has no poles to the right of the line. Then*

$$\sup\{f(z) \mid \Re(z) \geq 0\} = \sup\{f(z) \mid \Re(z) = 0\}.$$

Proof. Composing with a Möbius transformation we may instead consider the case of a rational function f holomorphic in the interior of the unit disk $\mathbb{D}$ and bounded on $\partial \mathbb{D} \setminus \{1\}$. The singularity of f at $z = 1$ is at most a pole since f is rational. The boundedness on the rest of the boundary then shows the singularity is removable so that f is continuous on the closed disk. Finally, apply the usual maximum principle. $\qquad\square$

Theorem 13. *Let $w \in W$, and let $\tilde{A}(\nu; w) \colon (I_\nu, V_K) \to (I_{w\nu}, V_K)$ be the intertwining operator, normalized such that $\tilde{A}(\nu; w)\varphi_0 = \varphi_0$. Then $\left\| \tilde{A}(\nu; w) \right\|_{L^2(K)} \leq 1$ for $\nu \in \bar{\Omega}_w$.*

Proof. By duality, it suffices to show that

$$\left\langle \tilde{A}(\nu; w)\varphi, \psi \right\rangle_{L^2(K)} \leq \|\varphi\|_{L^2(K)} \|\psi\|_{L^2(K)}$$

holds for all non-zero $\varphi, \psi \in V_K$ and all ν as above. As the left-hand-side is a meromorphic function of ν, it suffices to establish the inequality for $\Re(\nu) \in \mathscr{C}_w$, which we assume henceforth.

We restrict the left-hand-side to a one-parameter family of spectral parameters by considering the meromorphic one-variable function

$$f(z) = \frac{1}{\|\varphi\|_{L^2(K)} \|\psi\|_{L^2(K)}} \left\langle \tilde{A}(i\Im(\nu) + z\Re(\nu); w)\varphi, \psi \right\rangle_{L^2(K)}.$$

It will be convenient to write $\nu_z = i\Im(\nu) + z\Re(\nu)$ so that $\nu_1 = \nu$, and note that the parameters in our family satisfy $\Re(\nu_z) = \Re(z)\Re(\nu)$ and in particular $\Re(\nu_z) \in \mathscr{C}_w$ when $\Re(z) > 0$. Arthur's result quoted above (Fact 10(4)) is that $f(z)$ is a rational function of z. It has no poles in the domain $\Re(z) > 0$ since the intertwining operator has no poles in Ω_w. When $z = it \in i\mathbb{R}$, the parameter $\nu_z \in i\mathfrak{a}_{\mathbb{R}}^*$ is unitary and hence $\tilde{A}(\nu_z; w)$ is a unitary operator, which implies $|f(z)| \leq 1$ by Cauchy-Schwartz. In particular, f has no poles on this line, and the claim now follows from the Lemma. $\qquad\square$

Proof of Theorem 2. Let $(\pi, V_\pi) \in \hat{G}$ be spherical, and let $R\colon (I_\nu, V_K) \to (\pi, V_\pi)$ be a non-zero intertwining operator with the real part of $\nu \in \mathfrak{a}_\mathbb{C}^*$ in the closed positive chamber $\mathcal{C}$, normalized such that $\|R(\varphi_0)\|_{V_\pi} = 1$.

By Fact 10(3) there exists an involution $w \in W$ such that $w\nu = -\bar{\nu}$ and such that $\langle \varphi, A(\nu; w)\psi \rangle_{L^2(K)}$ is a G-equivariant Hermitian pairing on $(\mathcal{I}_\nu, V_K)$. Also, the image of $A(w, \nu)$ is irreducible (in fact, isomorphic to π). By Schur's Lemma there is $c \geq 0$ such that for all K-finite φ we have $\|R(\varphi)\|_{V_\pi} = c \langle \tilde{A}(\nu; w)\varphi, \varphi \rangle_{L^2(K)}$. Our normalization implies that the constant of proportionality is 1, and the bound on the intertwining operator gives the claim $\|R(\varphi)\|_{V_\pi} \leq \|\varphi\|_{L^2(K)}$.

3.1. Example: $\mathrm{SL}_2(\mathbb{R})$.

Let $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}_2(\mathbb{R})$. The Lie algebra $\mathfrak{g} =_2 (\mathbb{R})$ is spanned by the three elements $H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ and $\bar{X} = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}$. Individually they span the subalgebras $\mathfrak{a} = \mathbb{R}H$, $\mathfrak{n} = \mathbb{R}X$ and $\bar{\mathfrak{n}} = \mathbb{R}\bar{X}$. These are the Lie algebras of the subgroups $A = \left\{ \begin{pmatrix} * & \\ & * \end{pmatrix} \right\}$, $N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\}$, $\bar{N} = \left\{ \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix} \right\}$.

We shall also use $M = Z_K(A) = \{\pm I\}$ and fix $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, a representative for the non-trivial class in $W(\mathfrak{g} : \mathfrak{a}) \simeq N_K(A)/Z_K(A)$. Letting $k_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ so that $K = \{k_\phi\}_{\phi \in \mathbb{R}/2\pi\mathbb{Z}}$, we normalize the Haar measures on the circles K and $M\backslash K$ to be probability measures, on $\bar{N}$ to be $\frac{1}{\pi}du$ where $\bar{n}(u) = \exp(u\bar{X})$.

As $[H, X] = 2X$, we have $[tH, X] = \alpha(tH)X$ for that $\alpha \in \mathfrak{a}_\mathbb{R}^*$ (the "positive root") given by $\alpha(tH) = 2t$. We then set $\rho(tH) = \frac{1}{2}\alpha(tH) = t$ ("half the sum of the positive roots"). We can then identify the complex dual $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ via $z \mapsto (tH \mapsto zt)$.

The induced representation $\mathscr{P}^{+,z}$ (cf [8, §§2.5 & 7.1]) is the right regular representation of G on the space

$$\mathcal{F}^{+,z} = \left\{ F \in C^\infty(G) \mid F(n\exp(tH)mg) = e^{(z+1)t}F(g) \right\}.$$

By the Iwasawa decomposition these functions are uniquely determined by their restriction to the space $V = C(M\backslash K)$. the space of even functions on the circle. As usual we shall restrict our attention to the subspace $V_K \subset V$ of even trigonometric polynomials (the "K-finite" vectors), which is spanned by the Fourier modes $\varphi_{2m}(\theta) = \exp(2mi\theta)$.

As we will see shortly, for $\Re(z) > 0$ and $F \in \mathcal{F}^{+,z}$ the integral $(AF)(g) = \int_{\bar{N}} F(\bar{n}wg)d\bar{n}$ converges absolutely. Assuming this, we now verify that it defines an element of $\mathcal{F}^{+,-z}$. It also clearly intertwines the right regular representations under consideration.

For $a = \exp(tH) \in A$ we note that $waw^{-1} = a^{-1}$ and that for $\bar{n}' = a\bar{n}a^{-1} \in \bar{N}$ we have $d\bar{n}' = e^{-2t}d\bar{n}$. From this we conclude:

$$AF(ag) = \int_{\bar{N}} F(\bar{n}wag)d\bar{n} = e^{(-z+1)t} \int_{\bar{N}} F(\bar{n}'wg)d\bar{n}'.$$

Similarly we note that for $n \in N$, $wnw^{-1} \in \bar{N}$. Since we are integrating w.r.t. to a Haar measure on $\bar{N}$, this shows that $AF(ng) = AF(g)$. Finally, since M is central it is clear that $AF(mg) = AF(g)$. The smoothness is clear by differentiating under the integral sign, and the operator preserves K-finiteness since it commutes with the action of K.

Given $u \in \mathbb{R}$ we set $t = -\frac{1}{2}\log(1 + u^2)$ and define $\theta \in (0, \pi)$ by $u = \cot\theta$. Then there exists $n \in \mathbb{R}$ such that:

$$\bar{n}(u)w = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} \exp(tH)k_\theta.$$

Since $du = -\frac{d\theta}{\sin^2\theta}$ and $e^t = |\sin\theta|$ we get:

(3.1) $$A(z)F(k_\phi) = \frac{1}{\pi}\int_0^\pi |\sin\theta|^{z-1} F(k_{\phi+\theta})d\theta.$$

Since $F\upharpoonright_K$ is even this is a convolution operator on the circle K.

We can now address the question of convergence. Taking absolute values and bounding $F(k_{\phi+\vartheta})$ by $\|F\|_{L^\infty(K)}$ it is clear that $A(z)$ converges absolutely for all $F \in C(M\backslash K)$ iff the same holds for $A(\mathfrak{R}(z))\varphi_0$ where φ_0 is the constant function. When $F\upharpoonright_K$ is one of the Fourier modes φ_{2m}, we find on page 8 of [10] that the integral (3.1) converges absolutely for $\mathfrak{R}(z) > 0$ (the "open positive Weyl chamber") and takes the value:

$$A\varphi_{2m} = (-1)^m \frac{2^{1-z}\Gamma(z)}{\Gamma(\frac{z+1}{2}+m)\Gamma(\frac{z+1}{2}-m)} \cdot \varphi_{2m}.$$

We may thus extend $A(z)$ to a family of operators $A(z)\colon V_K \to V_K$ intertwining the induced representations and defined everywhere except for the pole at $z = 0$. We next normalize these operators. As above we define $r(z)$ by $A(z)\varphi_0 = r(z)\varphi_0$, that is:

$$r(z) = \frac{2^{1-z}\Gamma(z)}{\Gamma(\frac{z+1}{2})^2}.$$

Note that this meromorphic function has no zeroes or poles for $\mathfrak{R}(z) > 0$. In particular, if we set $\tilde{A}(z)F = r(z)^{-1}A(z)F$ the new operator is also regular for $\mathfrak{R}(z) > 0$ and extends meromorphically to $\mathbb{C}$. It will now have poles for $\mathfrak{R}(z) < 0$, but will be regular for $\mathfrak{R}(z) = 0$.

The claim of Theorem 2 (in this case) is that $\tilde{A}(z)\colon V_K \to V_K$ is bounded in the L^2 norm. Since it is diagonal in the Fourier basis it suffices to verify that the Fourier coefficients $\tilde{A}(z)\varphi_{2m} = c_{2m}(z)\varphi_{2m}$ satisfy $|c_{2m}(z)| \le 1$ when $\mathfrak{R}(z) \ge 0$.

For $m = 0$ this is true by definition of $r(z)$. In general, using $\Gamma(z+m) = \Gamma(z)\prod_{j=0}^{m-1}(z+j)$ we get:

$$c = \frac{\Gamma(\frac{z+1}{2})^2}{\Gamma(\frac{z+1}{2}+m)\Gamma(\frac{z+1}{2}-m)} = \prod_{j=0}^{m-1}\frac{z-(2j+1)}{z+(2j+1)}.$$

Now $z - (2j+1)$ and $z + (2j+1)$ always have the same imaginary part, but for $\mathfrak{R}(z) \ge 0$ the denominator always has a larger real part (in absolute value), and the product has magnitude at most 1 as claimed.

Remark 14. The rationality of the matrix coefficient $c_{2m}(z) = \bigl(\tilde{A}(z)\varphi_{2m}, \varphi_{2m}\bigr)_{L^2(M\backslash K)}$ was an essential ingredient in our argument above.

Corollary 15. *The normalized operator has no poles (or zeroes) for $\mathfrak{R}(v)$ in the* closed *positive chamber.*

4. DEGENERATE LIFT

In this section we establish Theorem 1.

4.1. **The basic construction.**

One eigenfunction. Let $\psi \in L^2(Y)$ be a normalized eigenfunction with the parameter $\nu \in \bar{\Omega}$; let $R\colon (\mathcal{I}_\nu, V_K) \to (\mathcal{R}, L^2(X)_K)$ be an intertwining operator with $R(\varphi_0) = \psi$. Given $f_1, f_2 \in V_K$ and $g \in C_c^\infty(X)_K$ we set:

$$\mu_R(f_1, f_2)(g) \overset{\text{def}}{=} \int_X R(f_1)\overline{R(f_2)}g\, d\,\mathrm{vol}_X\,.$$

By the Cauchy-Schwartz inequality and Theorem 2,

$$|\mu_n(f_1, f_2)(g)| \leq \|f_1\|_{L^2(K)} \|f_2\|_{L^2(K)} \|g\|_{L^\infty(X)}\,.$$

In particular, the $\mu_n(f_1, f_2)$ extend to finite Borel measures on X (positive measures when $f_1 = f_2$). Also, we have a bound on the total variation of these measures which depends only on f_1 and f_2 but not on ν or R.

This construction extends to the case where one of the two test vectors is not K-finite. Given $\Phi = \sum_{\tau \in \hat{K}} \phi_\tau \in \hat{V}_K$ we set:

$$\mu_R(f, \Phi)(g) = \sum_{\tau \in \hat{K}} \mu_R(f, \phi_\tau)(g),$$

noting that only finitely many τ can contribute. Letting $C_c^\infty(X)'_K$ denote the algebraic dual of $C_c^\infty(X)_K$, we have obtained a map :

$$\mu_R\colon V_K \times \hat{V}_K \to C_c^\infty(X)'_K$$

which is linear in the first variable and conjugate-linear in the second. Integration by parts on $\Gamma \backslash G$ shows that the extension $\mu_R\colon (\mathcal{I}_\nu, \mathcal{V}_K) \to C_c^\infty(X)'_K$ is an intertwining operator for the $(\mathfrak{g}, K)$ module structures.

Remark 16. By $C_c^\infty(X)'_K$ we mean the *algebraic* dual of our space $C_c^\infty(X)_K$ of test functions. By abuse of terminology we shall call its elements *distributions*; convergence of distributions will be in the weak-* (pointwise) sense. Apart from limits of uniformly bounded sequences of measures, the limits we shall consider will be *positive* distributions (that is, take non-negative values at non-negative test functions), and such distributions are always Borel measures (finiteness will require an easy separate argument). For completeness we note, however, that when Φ defines a distribution on $M\backslash K$ in the ordinary sense (as is the case with δ), $\mu_R(f, \Phi)$ is bounded w.r.t. to an appropriate Sobolev norm and hence $\mu_R(f, \Phi)$ is a distribution on X in the ordinary sense. Moreover, the bound depends on f and on the dual Sobolev norm of Φ but not ν or R.

A sequence of eigenfunctions. Let $\{\nu_n\}_{n=1}^\infty \subset \bar{\Omega}$ such that $\|\nu_n\| \to \infty$, and let $R_n\colon (\mathcal{I}_{\nu_n}, V_K) \to (\mathcal{R}, L^2(X)_K)$ be intertwining operators with $\|R_n(\varphi_0)\|_{L^2(X)} = 1$. Assume that $\bar{\mu}_n = \mu_n(\varphi_0, \varphi_0)$ converge weak-* to a limiting measure $\bar{\mu}_\infty$, which we would like to study.

Fixing $f_1, f_2 \in V_K$ the construction of the previous section gives a sequence of Borel measures $\mu_n(f_1, f_2) = \mu_{R_n}(f_1, f_2)$ all of which have total variation at most $\|f_1\|_{L^2(K)} \|f_2\|_{L^2(K)}$. By the Banach-Alaoglu theorem there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $\mu_{n_k}(f_1, f_2)$ converge weak-*. Fixing a countable basis $\{\varphi_i\}_{i=1}^\infty \subset V_K$, by the standard diagonalization argument we may assume (after passing to a subsequence) that for any $f_1, f_2 \in V_K$ there exists a measure $\mu_\infty(f_1, f_2)$ such that for all $g \in C_c^\infty(X)_K$,

$$\lim_{n \to \infty} \mu_n(f_1, f_2)(g) = \mu_\infty(f_1, f_2)(g).$$

As before, given f_1 and g, the value of $\mu_\infty(f_1, f_2)(g)$ only depends on the projection of f_2 to a finite set of K-types. We can thus extend μ_∞ to all of $\mathcal{V}_K = V_K \otimes \hat{V}_K$ and it is clear that μ_n converge weak-* to μ_∞ in the sense that for any fixed $F \in \mathcal{V}_K$ and $g \in C_c^\infty(X)_K$, $\lim_{n \to \infty} \mu_n(F)(g) = \mu_\infty(F)(g)$.

The asymptotic properties of μ_n are governed by the normalized spectral parameters $\tilde{\nu}_n = \frac{\nu_n}{\|\nu_n\|}$; passing to a subsequence again we assume $\tilde{\nu}_n \to \tilde{\nu}_\infty$ as $n \to \infty$. Since the $\mathfrak{R}(\nu_n)$ are uniformly bounded (we are dealing with unitary representations), the limit parameter $\tilde{\nu}_\infty$ is purely imaginary.

Definition 17. Call the sequence of intertwining operators $\{R_n\}_{n=1}^\infty$ *conveniently arranged* if $\tilde{\nu}_n$ converge to some $\tilde{\nu}_\infty \in i\mathfrak{a}_\mathbb{R}^*$ and if for any $f_1, f_2 \in V_K$ the sequence of measures $\{\mu_n(f_1, f_2)\}_{n=1}^\infty$ converges in the weak-* topology.

Given our limiting measure $\bar{\mu}_\infty$ we now fix once and for all a conveniently arranged sequence R_n such that $\mu_n(\varphi_0, \varphi_0)$ converges to $\bar{\mu}_\infty$, and set $M_1 = Z_K(\tilde{\nu}_\infty)$. The motivation for the following choice will be come clear in the following Section.

Definition 18. Let $\delta_1 \in V_K'$ be the distribution $\delta_1(f) = \int_{M \backslash M_1} f(m_1) dm_1$. Set:

$$\mu_n = \mu_n(\varphi_0 \otimes \delta_1),$$

which converge to the limit $\mu_\infty = \mu_\infty(\varphi_0 \otimes \delta_1)$.

Note that for a K-invariant test function g, $\mu_n(g) = \bar{\mu}_n(g)$ since the spherical part of δ_1 is exactly φ_0. It follows that the μ_n indeed are lifts of the measures $\bar{\mu}_n$ to $X = \Gamma \backslash G$, which is Claim (1) of the main Theorem.

Remark 19. Note that our definition of μ_n (and hence μ_∞) depends on the limit point $\tilde{\nu}_\infty$, and not only on the limiting measure $\bar{\mu}_\infty$.

4.2. **Integration by parts; positivity.** Pointwise addition and multiplication give an algebra structure to V_K. Our asymptotic calculus for the measures $\mu_n(f_1, f_2)$ will depend on the the following elements of this algebra.

For $X \in \mathfrak{g}$ and $k \in K$ we write the Iwasawa decomposition of $\mathrm{Ad}(k)X$ as $X_\mathfrak{n}(k) + X_\mathfrak{a}(k) + X_\mathfrak{k}(k)$. Now for $X \in \mathfrak{g}$ and $\tilde{\nu}_\infty \in i\mathfrak{a}_\mathbb{R}^*$ set:

$$p_X(k) = \frac{1}{i} \langle X_\mathfrak{a}(k), \tilde{\nu}_\infty \rangle$$

this is a left-M_1-invariant function on K, in particular a left-M-invariant function on K. It is K-finite, being a matrix element of the adjoint representation of K on $\mathfrak{g}$.

Lemma 20. *The subalgebra of V_K generated by $\{\varphi_0\} \cup \{p_X\}_{X \in \mathfrak{g}}$ under pointwise addition and multiplication is precisely $\mathcal{F}_1 = C(M_1 \backslash K)_K$, the algebra of left-$M_1$ invariant, right K-finite functions on K.*

Proof. This follows from the Stone-Weierstrass Theorem, by which it suffices to check that the functions p_X separate the points of $M_1 \backslash K$. Indeed, if $p_X(k) = p_X(k')$ for all X then $M_1 k' = M_1 k$ – recall that M_1 was defined as the centralizer of $\tilde{\nu}_\infty$. $\qquad\square$

Our calculation depends on the following basic formula, obtained by integration by parts:

Lemma 21. *([14, Lem. 3.10 & Cor. 3.11]) There exists a norm $\|\cdot\|$ on $C_c^\infty(X)_K$ such that for any $f_1, f_2 \in V_K$ and $X \in \mathfrak{g}$,*

$$\left| \mu_n(p_X f_1, f_2)(g) - \mu_n(f_1, \overline{p_X} f_2)(g) \right| \ll_{f_1, f_2} \|g\| \left[\|\tilde{\nu}_n - \tilde{\nu}_\infty\| + \|\nu_n\|^{-1} \right].$$

Corollary 22. *Let $f \in \mathcal{F}_1$ and $f_1, f_2 \in V_K$. Then, for any $g \in C_c^\infty(X)_K$,*

$$\left| \mu_n(f \cdot f_1, f_2)(g) - \mu_n(f_1, \overline{f} \cdot f_2)(g) \right| \ll_{f, f_1, f_2} \|g\| \left[\|\tilde{\nu}_n - \tilde{\nu}_\infty\| + \|\nu_n\|^{-1} \right].$$

Claims (2) and (4) of the main Theorem now follow from:

Proposition 23. *We can choose $f_n \in V_K$ (in the notation of the main Theorem, set $\tilde{\psi}_n = R_n(f_n)$) so that the measures $\sigma_n = \mu_n(f_n, f_n)$ converge weak-* to μ_∞.*

Proof. Let $\{h_k\}_{k=1}^\infty \in \mathcal{F}_1$ be real-valued functions such that h_k^2 converge weak-* to δ_1, and let $h_0 = \varphi_0$ (it is easy to see that such a sequence exists). By Corollary 22 there exists constants C_k depending only on the choice of f_k such that for any $g \in C_c^\infty(X)_K$ and n,

$$\left| \mu_n\left(\varphi_0, h_k^2\right)(g) - \mu_n(h_k, h_k)(g) \right| \le C_k \|g\| \left[\|\tilde{v}_n - \tilde{v}_\infty\| + \|v_n\|^{-1} \right].$$

Noting that $C_0 = 0$, given $n \ge 1$ let $k(n)$ be the maximal $k \in \{0, \cdots, n\}$ such that $C_k \le \left[\|\tilde{v}_n - \tilde{v}_\infty\| + \|v_n\|^{-1} \right]^{-1/2}$, and set $f_n = h_{k(n)}$, $\sigma_n = \mu_n(f_n, f_n)$. The sequence $k(n)$ is monotone and tends to infinity; it follows that f_n^2 converge weakly to δ_1.

Finally, we have:

$$|\mu_n(g) - \sigma_n(g)| \le \left| \mu_n\left(\varphi_0, \delta_1 - f_n^2\right)(g) \right| + \left[\|\tilde{v}_n - \tilde{v}_\infty\| + \|v_n\|^{-1} \right]^{1/2} \|g\|.$$

Let $T \subset \hat{K}$ be a finite subset such that $g \in \sum_{\tau \in T} C_c^\infty(X)_\tau$. Let $d_n \in \sum_{\tau \in T} V_\tau$ be the projection of $\delta_1 - f_n^2$ to that space. Then $R(\delta_1 - f_n^2 - d_n)$ has trivial pairing with $R(\varphi_0)g$, since they don't transform under the same K-types. We may thus bound the first term in the inequality above by $|\mu_n(\varphi_0, d_n)(g)| \le \|d_n\|_{L^2(K)} \|g\|_{L^\infty(X)}$. Since $\sum_{\tau \in T} V_\tau$ is finite-dimensional, that $d_n \to 0$ weakly implies that $d_n \to 0$ in norm. Since $\mu_n(g) \to \mu_\infty(g)$ we conclude that $\sigma_n(g) \to \mu_\infty(g)$ as well. $\square$

Corollary 24. *μ_∞ extends to a non-negative measure on X of total mass at most 1. When X is compact μ_∞ is a probability measure.*

Proof. The σ_n extend to positive measures, hence μ_∞ extends to a non-negative measure. To bound the total mass it suffices to consider K-invariant test functions for which μ_∞ agrees with $\bar{\mu}_\infty$, a weak-* limit of probability measures. $\square$

Corollary 25. *When ψ_n are eigenfunctions of an algebra $\mathcal{H}$ of operators which commute with the G-action, then so are $\tilde{\psi}_n$.*

Proof. By Schur's Lemma each element of $\mathcal{H}$ acts as a scalar on the irreducible representation generated by ψ_n; $\tilde{\psi}_n$ belongs to this representation. $\square$

4.3. A_1-**invariance.** Let $\delta \in V_K' \simeq \hat{V}_K$ be the delta distribution, that is $\delta(f) = f(1)$. Since $\mathfrak{a}$ is a quotient of $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ by a Lie ideal, we can consider any $\lambda \in \mathfrak{a}_\mathbb{C}^*$ as a Lie algebra homomorphism $\mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C} \to \mathbb{C}$. It thus extends to an algebra homomorphism $U(\mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C}) \to \mathbb{C}$, and there exists a unique algebra endomorphism $\tau_\lambda : U(\mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C}) \to U(\mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C})$ such that $\tau_\lambda(X) = X + \lambda(X)$ for $X \in \mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C}$.

Lemma 26. *Let $v \in \mathfrak{a}_\mathbb{C}^*$, $u \in U(\mathfrak{a}_\mathbb{C} \oplus \mathfrak{m}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C})$*
    (1) $I_v(u)\delta = (-\rho + v)(u) \cdot \delta$.
    (2) $\mathcal{I}_v(\tau_{\rho + v - 2\Re(v)}(u))(f \otimes \delta) = (I_v(u)f) \otimes \delta$.

Proof. By induction it suffices to prove both assertions for $u = X \in \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. The first claim follows from the the invariant pairing of $\mathcal{F}^v$ with $\mathcal{F}^{-v}$. Taking complex conjugates, this implies:

$$\mathcal{I}_v(X)(f \otimes \delta) = (I_v(X)f) \otimes \delta + \langle -\rho_\mathfrak{a} + \bar{v}, X \rangle (f \otimes \delta),$$

which is the second assertion. $\square$

We next summarize the analysis of the center of the universal enveloping algebra done in [14, §4].

Proposition 27. *Let $\mathcal{P} \in U(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}$ be homogeneous of degree d. Then there exist elements $b = b(\mathcal{P}) \in U(\mathfrak{n}_{\mathbb{C}})U(\mathfrak{a}_{\mathbb{C}})^{\leq d-2}$ and $c = c(\mathcal{P}) \in U(\mathfrak{g}_{\mathbb{C}})\mathfrak{k}_{\mathbb{C}}$ so that*

$$z = \tau_{-\rho_{\mathfrak{h}}}(\mathcal{P}) + b + c$$

belongs to the center of the universal enveloping algebra. Furthermore, z acts on (I_{ν}, V_K) with the eigenvalue $\mathcal{P}(\nu + \rho_{\mathfrak{a}} - \rho_{\mathfrak{h}})$.

It follows that for such $\mathcal{P}$ we have:

$$\mathcal{I}_{\nu}\left(\tau_{\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}+\nu-2\Re(\nu)}(\mathcal{P}) + \tau_{\rho_{\mathfrak{a}}+\nu-2\Re(\nu)}(b) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}})\right)(\varphi_0 \otimes \delta) = 0$$

(to see this unwind the definitions, using the fact that $I_{\nu}(c)\varphi_0 = 0$).

Thinking of $\mathcal{P}$ as a function on $\mathfrak{h}_{\mathbb{C}}^*$, let $P'(\nu)$ denote its differential at $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. This is an element of the cotangent space to $\mathfrak{h}_{\mathbb{C}}^*$, that is an element of $\mathfrak{h}_{\mathbb{C}}$.

Proposition 28. *Let $\mathcal{P} \in U(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}$. Then there exists a polynomial map $J: \mathfrak{a}_{\mathbb{C}}^* \to U(\mathfrak{g}_{\mathbb{C}})$ ($\mathfrak{a}_{\mathbb{C}}^*$ thought of as a real vector space), of degree at most $d-2$ in the parameters $\Im(\nu)$, such that for any unitarizable parameter $\nu \in \mathfrak{a}_{\mathbb{C}}^*$,*

$$\mathcal{I}_{\nu}\left(\mathcal{P}'(\tilde{\nu}) + \frac{J(\nu)}{\|\nu\|^{d-1}}\right)(\varphi_0 \otimes \delta) = 0.$$

Proof. Since $\mathcal{P}'$ is a homogeneous polynomial of degree $d-1$, it suffices to show that $\tau_{\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}+\nu-2\Re(\nu)}(\mathcal{P}) + \tau_{\rho_{\mathfrak{a}}+\nu-2\Re(\nu)}(b) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}) - \mathcal{P}'(\nu)$ is a polynomial of degree at most $d-2$ in ν. It is clear that $J_1(\nu) = \tau_{\rho_{\mathfrak{a}}+\nu-2\Re(\nu)}(b)$ is such a polynomial, as is $J_2(\nu) = \tau_{\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}+\nu-2\Re(\nu)}(\mathcal{P}) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\Re(\nu)) - \mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\Re(\nu))$. Since $\mathcal{P}'$ is a polynomial of degree $d-1$ (valued in $\mathfrak{a}_{\mathbb{C}}$), $J_3(\nu) = \mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\Re(\nu)) - P'(\nu)$ is also of degree at most $d-2$. It remains to consider $J_4(\nu) = \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}-2\Re(\nu)) - \mathcal{P}(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}})$ which is a polynomial map of degree $d-1$ in ν.

The first two terms are difference of the values of a polynomial at two points, we may write this in the form $\langle\mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}), -2\Re(\nu)\rangle + J_5(\nu)$ where $J_5(\nu)$ includes the terms of degree $d-2$ or less and the pairing is the one between $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{a}_{\mathbb{C}}^*$. Finally, $\mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}) - \mathcal{P}'(\Im(\nu))$ has degree $d-2$ in $\Im(\nu)$. Setting $J_6(\nu) = \langle\mathcal{P}'(\nu+\rho_{\mathfrak{a}}-\rho_{\mathfrak{h}}) - \mathcal{P}'(\Im(\nu)), -2\Re(\nu)\rangle$ we see that $\varphi_0 \otimes \delta$ is annihilated by:

$$P'(\nu) + \sum_{i=1}^{6} J_i(\nu) - 2\langle\mathcal{P}'(\Im(\nu)), \Re(\nu)\rangle.$$

We conclude by showing that the final scalar vanishes. By assumption there exists $w \in W$ such that $w\nu = -\bar{\nu}$. By Corollary 6 there exist $\tilde{w} \in W(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$ such that $\tilde{w}\nu = -\bar{\nu}$. Applying the chain rule to $\mathcal{P} = \mathcal{P} \circ \tilde{w}$ we see that $\mathcal{P}'(\Im(\nu))$ is fixed by $\tilde{w}$, while $\tilde{w}\Re(\nu) = -\tilde{w}\Re(\nu)$. $\square$

Corollary 29. *Let $\{R_n\}_{n=1}^{\infty}$ be a conveniently arranged sequence of intertwining operators from (I_{ν_n}, V_K) to $L^2(X)$. Then the limit distribution $\mu_{\infty} = \mu_{\infty}(\varphi_0 \otimes \delta_1)$ is H-invariant for any $H = \mathcal{P}'(\tilde{\nu}_{\infty})$, where $\mathcal{P} \in U(\mathfrak{h}_{\mathbb{C}})^{W(\mathfrak{g}_{\mathbb{C}}:\mathfrak{h}_{\mathbb{C}})}$.*

Proof. By additivity it suffices to prove this when $\mathcal{P}$ is homogeneous. Next, since $\mu_n = \mu_n(\varphi_0 \otimes \delta_1)$ are M_1-invariant distributions (in fact, we are lifting to $\Gamma\backslash G/M_1$, not to $\Gamma\backslash G$), it suffices to consider M_1-invariant test functions $g \in C_c^{\infty}(X)_K^{M_1}$. For these we have $\mu_n(\varphi_0 \otimes \delta_1)(g) =$

$\mu_n (\varphi_0 \otimes \delta)(g)$. We conclude that it is enough to show that $\mu_\infty (\varphi_0 \otimes \delta)$ are $\mathcal{P}'(\tilde{v}_\infty)$-invariant for homogeneous $\mathcal{P}$ – but this follows immediately by passing to the limit in the Proposition. $\qquad\square$

Since the $W(\mathfrak{g}_\mathbb{C} : \mathfrak{h}_\mathbb{C})$-invariant polynomials on $\mathfrak{h}_\mathbb{C}$ are dense in the space of smooth functions on the sphere there, it is clear that $\{\mathcal{P}'(\tilde{v}_\infty)\}$ is precisely the set $\mathfrak{h}_\mathbb{C}^{W'_1}$ where $W'_1 = \mathrm{Stab}_{W(\mathfrak{g}_\mathbb{C}:\mathfrak{h}_\mathbb{C})}(\tilde{v}_\infty)$. Claim (3) of the main Theorem is then contained in:

Lemma 30. *Let $W_1 = \mathrm{Stab}_W(\tilde{v}_\infty)$. Then $\mathfrak{a}_\mathbb{C}^{W_1} = \mathfrak{h}_\mathbb{C}^{W'_1} \cap \mathfrak{a}_\mathbb{C}$.*

Proof. The subgroup of a Weyl group fixing a point in $\mathfrak{a}$ (or its dual) is generated by the root reflections it contains. It follows that W_1 is generated by the root reflections s_α where $\alpha \in \Delta(\mathfrak{g} : \mathfrak{a})$ satisfying $B(\alpha, \tilde{v}_\infty) = 0$ (pairing given by the Killing form on $\mathfrak{g}$) while W'_1 is generated by the root reflections $s_{\alpha'}$ where $\alpha' \in \Delta(\mathfrak{g}_\mathbb{C} : \mathfrak{h}_\mathbb{C})$ satisfies $B'(\alpha', \tilde{v}_\infty) = 0$ (Killing form on $\mathfrak{g}_\mathbb{C}$). Now $B'(\alpha', \tilde{v}_\infty) = B(\alpha' \restriction_\mathfrak{a}, \tilde{v}_\infty)$ since $\mathfrak{b}$ is orthogonal to $\mathfrak{a}$, where the restrictions are either roots (or zero). Since s_α fixes H iff $\alpha(H) = 0$, while $s_{\alpha'}$ fixes H iff $\alpha'(H) = 0$, it follows that:

$$\mathfrak{h}_\mathbb{C}^{W'_1} \cap \mathfrak{a}_\mathbb{C} = \cap_{B(\alpha,\tilde{v}_\infty)=0} \mathrm{Ker}(\alpha) = \mathfrak{a}_\mathbb{C}^{W_1}.$$

$\qquad\square$

REFERENCES

[1] Nalini Anantharaman. Entropy and the localization of eigenfunctions. *Ann. of Math. (2)*, 168(2):435–475, 2008.

[2] Nalini Anantharaman and Lior Silberman. A haar component for quantum limits on locally symmetric spaces. preprint, `arXiv:math.AP/1009.4927`, 2010.

[3] James Arthur. Intertwining operators and residues. I. Weighted characters. *J. Funct. Anal.*, 84(1):19–84, 1989.

[4] Jean Bourgain and Elon Lindenstrauss. Entropy of quantum limits. *Comm. Math. Phys.*, 233(1):153–171, 2003.

[5] Yves Colin de Verdière. Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.*, 102(3):497–502, 1985.

[6] Manfred Einsiedler, Anatole Katok, and Elon Lindenstrauss. Invariant measures and the set of exceptions to littlewood's conjecture. *Ann. of Math. (2)*, 164(2):513–560, 2006.

[7] A. W. Knapp and E. M. Stein. Intertwining operators for semisimple groups. *Ann. of Math. (2)*, 93:489–578, 1971.

[8] Anthony W. Knapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples.

[9] Elon Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2)*, 163(1):165–219, 2006.

[10] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni. *Formulas and theorems for the special functions of mathematical physics*. Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

[11] François Maucourant. A nonhomogeneous orbit closure of a diagonal subgroup. *Ann. of Math. (2)*, 171(1):557–570, 2010.

[12] Zeév Rudnick and Peter Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.*, 161(1):195–213, 1994.

[13] Lior Silberman and Akshay Venkatesh. Entropy bounds for Hecke eigenfunctions on division algebras. to appear in GAFA.

[14] Lior Silberman and Akshay Venkatesh. Quantum unique ergodicity for locally symmetric spaces. *Geom. Funct. Anal.*, 17(3):960–998, 2007. arXiv:math.RT/407413.

[15] A. I. Šnirel′man. Ergodic properties of eigenfunctions. *Uspekhi Mat. Nauk*, 29(6(180)):181–182, 1974.

[16] Kannan Soundararajan. Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash\mathbb{H}$. *Ann. of Math. (2)*, 172(2):1529–1538, 2010.

[17] Steven Zelditch. Pseudodifferential analysis on hyperbolic surfaces. *J. Funct. Anal.*, 68(1):72–105, 1986.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER BC V6T 1Z2, CANADA

E-mail address: lior@math.ubc.ca